Novel Initial Parameters Computation for EM algorithm-based Univariate Asymmetric Generalized Gaussian Mixture

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***Abstract*—In histogram-based image segmentation, the Asymmetric Generalized Mixture Model (AGGMM) is a powerful tool to fit accurately the real images histograms by handling, among others, any asymmetry of the modes. However, the Expectation Maximization (EM) algorithm, used for the estimation of the mixture model parameters, is known to be very sensitive to starting conditions and can lead to erroneous segmentation results when the initialization is not adequate. In this paper, we propose a new method to initialize the AGGMM. This method is based on geometrical aspects of the histogram. First experimentations implying synthetic images generated by Asymmetric Generalized Mixture Distribution (AGGD) model, reveal a good recovering of the input mixture parameters when applying the proposed method. Second experimentations involving real-world images have shown, how the initial parameters computed by the proposed method permit to achieve better histogram fitting with less EM algorithm running time in comparison to other initialization methods.**

***Index Terms*—Geometrical aspect, Asymmetric Generalized Gaussian, Mixture model, Initialization**

I. Introduction

Image segmentation is an important ongoing research of computer vision. It consists in dividing the image into smaller parts that are easier to analyse by further analysis stages. Image segmentation methods based on the image intensity probability distribution rely on nonparametric and parametric approaches. When exploiting the parametric approaches, the classes defining the various image partitions can be separated by considering a statistical model to approach the class probability density functions (*pdf*) [1]. When considering mixture model based-image segmentation with the aim to portray the data accurately, the asymmetry of the histogram modes must be handled. This can be achieved by making use of the Asymmetric Generalized Gaussian Mixture Model (AGGMM) which can approximate a large class of statistical distributions (e.g. impulsive, Laplacian, Gaussian and uniform) [1]. We recall that for the Expectation Maximization algorithm (EM) [2], an initialization step is required where all the mixture model parameters starting values should be introduced. The initialization plays an important role in the EM algorithm convergence in terms of speed and solution accuracy. On the other hand, the computation of the AGGMM parameters involves some highly non linear equations [1]. Solving such equations is done numerically and iteratively which involves additive computation cost. Consequentially, one should reduce the computation load by providing an appropriate initialization to speed up the convergence. In our knowledge, the issue of initializing all AGGMM parameters has not been tackled yet. Some attempts have been made to provide an initialization to probabilistic mixture models on the basis of geometrical considerations. In [3], the authors have proposed an initialization of Gaussian Mixture Models based on the histogram shape and the mode symmetry, but this work cannot be applicable to asymmetric distributions. The work in [4] proposes to compute the shape parameter and the variance of one GGD, which is not appropriate neither for a mixture, nor for an AGGD. To sum up, some AGGMM parameters initialization such as the mean values and the mixture proportions initialization can be found in the literature as they are shared by several distribution mixtures models. Nevertheless, the issue of providing an initialization for the shape factor and the left/right standard deviation for AGGMM has not been addressed yet. In this work, we propose an initial parameters computation for the EM algorithm-based AGGMM that relies exclusively on the geometrical aspect of the histogram.

The remainder of this paper is organized as follows. Section 2 consists in an overview of the AGGM model. Section 3 is devoted to the proposed initial parameters computation. The experimental results of the proposed method applied on histograms of synthetic and real images are presented in Section 4. In section 5, the main conclusions are drawn.

II. AGGMM and EM algorithm

The *pdf* of univariate AGGD defined for x ($x \in \mathbb{R}$) where $\theta = (\mu, \alpha_1, \alpha_2, \beta)^T$ ($\mu \in \mathbb{R}$, $\alpha_1, \alpha_2, \beta \in \mathbb{R}_+^*$) is the distribution parameter vector with the mean value parameter, the left and the right scale parameters and the shape parameter, respectively is given by:

$$f(x/\theta) = \begin{cases} \dfrac{\beta}{(\alpha_1+\alpha_2)\Gamma(1/\beta)} e^{-[(-x+\mu)/\alpha_1]^\beta} & \text{if} \quad x < \mu \\ \dfrac{\beta}{(\alpha_1+\alpha_2)\Gamma(1/\beta)} e^{-[(x-\mu)/\alpha_2]^\beta} & \text{if} \quad x \geq \mu \end{cases} \quad (1)$$

The gamma function is defined by $\Gamma(\zeta) = \int_0^\infty e^{-t} t^{\zeta-1} dt$. β determines whether the *pdf* is peaked or flat, α_i, i=1,2 is a scale factor linked to the standard deviation σ_i as following: $\alpha_i = \sigma_i \sqrt{\Gamma(1/\beta)/\Gamma(3/\beta)}$. For the purpose of image segmentation, let us assume that the normalized gray level histogram $h(g)$, $g \in [0, \ldots, G-1]$, with G the number of gray levels, can be approximated by an AGGMM of M components $f_{mix}(x/\Theta)$ as $f_{mix}(x/\Theta) = \sum_{m=1}^{M} \pi(m) f(x/\theta(m))$. Here, Θ=$(\pi(m), \theta(m))$ is the vector parameter to be estimated, given that, $m = 1, \ldots, M$, with $\pi(m)$ is the mixing parameter obeying to: $\pi(m) > 0$ and $\sum_m \pi(m)$=1, and $\theta(m) = (\mu(m)$, $\alpha_1(m)$, $\alpha_2(m)$, $\beta(m))$. As shown in [1], the complete log-likelihood function is given by

$$L_c(\Theta) = \sum_g \sum_m z(g,m) h(g) \log[\pi(m) f(g/\theta_m)] \quad (2)$$

Using the Maximum Likelihood (ML) estimation by means of EM algorithm, the posterior probabilities $z(g,m)$ and the mixture parameters estimates $\hat{\Theta}$ are provided in [1]. It is worth to note that all the derived equations in Maximization-step are highly nonlinear, and the numerical solutions issued from Newton–Raphson method could be unstable due, among others, to differentiation involved at the function breakpoint [1]. This instability arises, also, when providing the EM algorithm inadequate starting mixture parameters.

III. EM ALGORITHM INITIALIZATION FOR AGGMM

A. *Components number M and mean values μ initialization.*

The amount $\pi(m) f(g/\theta(m))$ must fit the m^{th} mode in the image histogram $h(g)$. Let us consider that image intensities *pdfs* peaks are the histogram modes peaks, as widely approved [5]. Detecting the peaks provides the initial mean parameters μ_{init} and counting them provides M. However, image histograms are not as smooth as a simple curve differentiation could detect the modes maxima. Moreover, excessive smoothing could shift the maxima. Consequently, as done in [5] for detecting true modes by detecting true minima, and for the sake of accuracy, we have detected maxima and minima using scale space. Minima will be used later. By definition, the local maximum (minimum) is the discrete point whose amplitude is greater (smaller) than the adjacent left and right points simultaneously. Nevertheless, $h(g)$ could present multiple points falling within this definition and the peaks should be found among these multiple points. To overcome such inconvenience, a scale space representation $L(g,s)$ is computed for $h(g)$ by convolving repetitively $h(g)$ with a Gaussian kernel $k(g,\sigma_k)$, where s is the scale space parameter and σ_k a standard deviation. This operation removes progressively local "patterns" from $h(g)$ while detecting, simultaneously, local maxima on the smoothed versions of $h(g)$. The number of

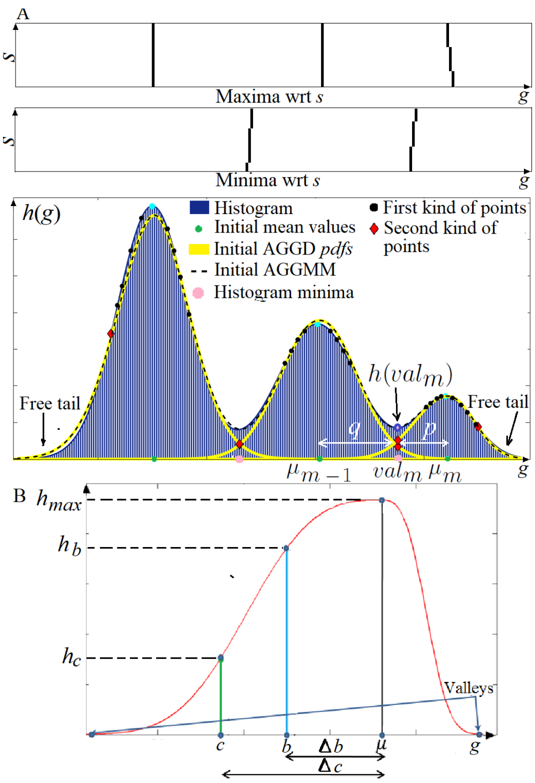

Fig. 1: A: From top to bottom: Scale space plane of the maxima, scale space plane of the minima, example of an histogram and its initial AGGD curves. B: AGGD Mode and an example of a couple of selected points of coordinates (b, h_b) and (c, h_c)

local maxima in the scale space $L(g,s)$ is decreasing with an increase of s. Assuming that the number of initial maxima $h(p_0)$ is N_0, then, when increasing s from 1 to S, where S is the number of scales, each of the initial maxima arguments $p_0(i)$, $i = 1, \ldots, N_0$, generates a curve $C(p_0(i))$, of a length equal to $l(i)$, in a scale space plane of dimensions $S \times G$, as seen in Fig. hist-curv.A. If the length $l(j)$ of a curve $C(p_0(j))$ exceeds a threshold T, then, $h(p_0(j))$ is a peak of $h(g)$ and $_0^p(j)$ is an initial mean parameter value. The same operation is done to compute the minima. Furthermore, T is computed as the mean value of all the generated curves lengths, i.e. $T = \sum_{i=1}^{N_0} l(i)/N_0$. However, when the image is very noisy, this operation must be preceded by a light smoothing to reduce noise in the histogram as done in [5].

B. *Mixing proportions π initialization*

Let $\mathbf{V} = [val(1), \ldots, val(N_V)]$ be the arguments of the N_V computed $h(g)$ minima (valleys). In order to locate modes, we examine these minima. The starting proportions are then equal to $\pi_{init}(m) = \sum_{val(m)}^{val(m+1)} h(g)$, where $m = 1, ..., M$

C. *Shape and scale factors β, α_1 and α_2 initialization*

Let us see, in the following, how $\alpha_{1,2}$ and β are derived geometrically from an AGGD mode. Let us note h_{max} the maximum of an AGGD mode and b and c the x-values of two points belonging to left side of the mode curve, as seen in Fig. hist-curv.B. Let us recall that for an AGGD mode $h_{max} = h(\mu)$, where μ is the mean parameter of this mode. Furthermore, let us note $h_b = h(b)$, $h_c = h(c)$. Using Eq.1, h_{max}, h_b and h_c are computed as $\beta/((\alpha_1 + \alpha_2)\Gamma(1/\beta))$, $h_{max}e^{-[(\mu-b)/\alpha_1]^\beta}$ and $h_{max}e^{-[(\mu-c)/\alpha_1]^\beta}$, respectively. By setting Δb=$\mu - b$ and Δc=$\mu - c$, we provide α_1 wrt [1] h_{max}, h_b, h_c, Δb and Δc:

$$\alpha_1 = e^{(\kappa \log \Delta c - \log \Delta b)/(\kappa - 1)} \quad \text{where} \quad \kappa = \log(-\log(h_b/h_{max}))/\log(-\log(h_c/h_{max})) \tag{3}$$

Proof:

By inspecting the AGG mode of Fig. hist-curv.B we can write:

$h_b/h_{max} = e^{-[\Delta b/\alpha_1]^\beta} \Rightarrow -\log(h_b/h_{max}) = [\Delta b/\alpha_1]^\beta$.

Knowing that, $h_b < h_{max}$, then $-\log(h_b/h_{max}) > 1$,

So, we can write $\log(-\log(h_b/h_{max})) = \beta \log(\Delta b/\alpha_1)$.

The same reasoning is used for $h_c/h_{max} = e^{-[\Delta c/\alpha_1]^\beta}$ to obtain $\log(-\log(h_c/h_{max})) = \beta \log(\Delta c/\alpha_1)$.

Thus, $\log(\Delta b/\alpha_1)/\log(\Delta c/\alpha_1) = \log(-\log(h_b/h_{max}))/\log(-\log(h_c/h_{h_max})) = \kappa$.

Consequently, $\log(\Delta b/\alpha_1) = \kappa \log(\Delta c/\alpha_1)$.

By developing, we have,

$\log \Delta b - \log \alpha_1 = \kappa \log \Delta c - \kappa \log \alpha_1$,

Which is equivalent to $(\kappa - 1) \log \alpha_1 = \kappa \log \Delta c - \log \Delta b$.

Implying finally, $\log \alpha_1 = (\kappa \log \Delta c - \log \Delta b)/(\kappa - 1)$, and then, $\alpha_1 = e^{(\kappa \log \Delta c - \log \Delta b)/(\kappa - 1)}$

On the other hand:

$$\beta = \log(-\log(h_c/h_{max}))(\Delta c/\alpha_1)^{-1} = \log(-\log(h_b/h_{max}))(\Delta b/\alpha_1)^{-1} \tag{4}$$

It is recommanded, while computing $\alpha_{1,2,init}$ and β_{init}, to select more than two points on each side of the considered mode and compute all the corresponding scale and shape factors. Furthermore, one of the selected points is chosen depending of this mode minima. This is why we have taken the following considerations: Depending on the overlapping of the histogram modes, two kinds of points are chosen to compute $\alpha_{1,2,init}$. The first kind consists in points selected on the upper part of both mode sides, as it is less affected by modes overlapping. The second kind of points are picked to refine the scale factor computed with the upper modes parts points. In this case, depending on the existence or not of mode overlapping, two configurations could happen. The first one occurs when there is no overlapping in the considered mode side and here, a point is selected at the mode half heigh of the involved side, i.e. if $c_{h/2}$ is the x-value of this point, then $h(c_{h/2}) = h_{max}/2$. The second configuration concerns the presence of overlapping and here, the point is chosen under the mode valley of the considered side. In order to develop and explain what was previously introduced, let us consider the left side of the m^{th} mode. Here, k values of $\alpha_{l,(i,j)}$, $i, j \in 1, \ldots, k$ with $i \neq j$ are computed with k couples of the first kind points, using Eq.3, where, l is for left side. These computed values are exploited in Eq.4 to compute k values of $\beta_{l,(i,j)}$, that will, in turn, be used to compute β_l as the median of these computed $\beta_{l,(i,j)}$. Subsequently, β_l is used to compute, differently, another value of a the left scale factor named α_{l,kd_2} with the second kind of points. It worth to note that we consider that there is no overlapping for the investigated mode side if $h(val(m))/h_{max} < 0.1$). In this case, α_{l,kd_2} is obtained by deriving its expression from Eq.4

$$\alpha_{l,kd_2} = (\mu - c_{h/2}) \log(h(c_{h/2})/h_{max})\beta_l^{-1} = (\mu - c_{h/2}) \log(1/2)\beta_l^{-1} \tag{5}$$

When there is an overlapping, the point under the mode valley is used as the second kind of points. The x-value of this point is $val(m)$ but its ordinate is unknown and must be computed. Besides involving the m^{th} mode left side, in the computation of this point ordinate, the $(m-1)^{th}$mode right side is also involved, since the sum of the two consecutive AGGDs modes at $g = val(m)$ equals $h(val(m))$. The ordinate of this point depends on $|val(m) - \mu(m-1)|$ and $|val(m) - \mu(m)|$, as illustrated in Fig. hist-curv.A. If the $|val(m) - \mu(m-1)| = q$ and $|val(m) - \mu(m)| = p$, then, the contribution the m^{th} mode left side to reach $h(val(m))$ equals to $q/(p+q)$ and then, the ordinate of this point is $y_{kd_2} = h(val(m))q/(p+q)$. It is obvious that the same reasoning is followed for the $(m-1)^{th}$ mode right side where its contribution is $p/(p+q)$. Here, α_{l,kd_2} is given by

$$\alpha_{l,kd_2} = (\mu(m) - val(m)) \log(y_{kd_2}/h_{max})\beta_l^{-1} \tag{6}$$

After these operations, we can compute $\alpha_{1,init}$ of the mode under consideration by

$$\alpha_{1,init} = (\alpha_{l,kd_2} + median(\alpha_{l,1}, \ldots, \alpha_{l,k}))/2 \tag{7}$$

The same procedure is applied to the right side of the mode to compute $\alpha_{2,init}$ and β_r. To finish, the shape factor initialization for the considered mode equals to

$$\beta_{init} = (\beta_l + \beta_r)/2. \tag{8}$$

D. AGGMM parameters initialization algorithm

Algorithm 1 summarizes our initialization method.

IV. EXPERIMENTS

The first test image and its histogram $h(g)$ are depicted on Fig. 2. The gray values distribution of this image follows the AGGD with means equal to [35 100 180] corresponding to the 3-mode peaks in $h(g)$. The computed initialization of the AGGMM parameters are given in Table.1 and the corresponding initial *pdf* is illustrated in Fig. 2. The estimated AGGMM parameters with EM algorithm for our initialization and the initialization in [1] are provided in table.2 and the related issued *pdfs*, noted $f(g)$ are also depicted in Fig. 2. We can see when inspecting visually Fig. 2, that there is a slight difference between our proposed initial AGGMM *pdf* and the *pdfs* issued from EM algorithm-based AGGMM estimation, whether initializing μ, $\alpha_{1,2}$ and β as done in [1] or initializing

[1] with respect to

Input: Image histogram $h(g)$

Output: $M, \pi_{init}, \mu_{init}, \alpha_{1,init}, \alpha_{2,init}, \beta_{init}$

Compute $h(g)$ modes peaks and valleys

Deduce μ_{init} and M

for $n = 1 : M$ **do**

  Select the n^{th}mode

  for *each side* **do**

    Select k couple of points in the upper part

    Compute with Eq.3 k values of $\alpha_{l/r,i,j}$

    Compute k values of $\beta_{l/r,i}$ with k values of $\alpha_{l,i,j}$ and Eq.4

    Compute $\beta_{l/r}$ as the median value of $\beta_{l/r,i}$, $i = 1, \ldots, k$

    Compute $\alpha_{l/r,kd_2}$ with Eq.5 or Eq.6

    Compute the $n^{th}\alpha_{1/2,init}$ with Eq.7

  end for

  Compute the $n^{th}\beta_{init}$ with Eq.8

end for

return $M, \pi_{init}, \mu_{init}, \alpha_{1,init}, \alpha_{2,init}, \beta_{init}$

Algorithm 1: AGGMM parameters initialization

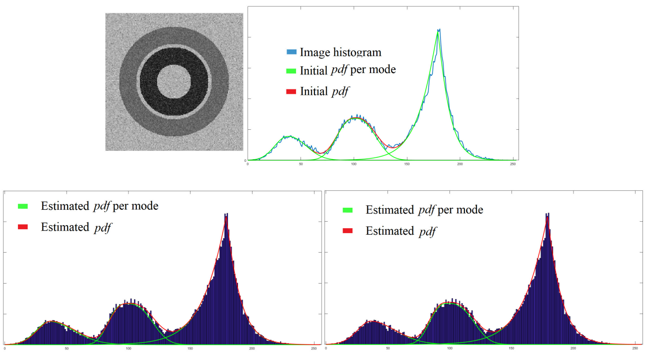

Fig. 2: Top: Synthetic image, its histogram and the computed initialization. Bottom: AGGMM results after EM convergence with initialization in [1] (left) and with our initialization (right)

them with our method. This fact suggests that the proposed initialization is very close to the final solution. This supposition is confirmed by the little gap between the computed initial parameters and the real ones (model design) expressed as absolute errors (Δ) in Table.1. Moreover, Table.2 shows that the number of iterations needed by the EM algorithm to converge to the final AGGMM is equal to 6 when using our initialization and 293 iterations when using the other initialization. On the other hand, the computed root-mean-square error $er_{fit} = \sqrt{\frac{1}{G} \times \sum_{g=1}^{G}(h(g) - f_{mix}(g))^2}$ between the image histogram and the computed AGGMM for both cases are substantially the same as shown in Table.2. In addition, We can notice in Table.1 that even if Δ values are small, there is a case where Δ is relatively important. Indeed, some parameters linked to the first mode present a quite important deviations to the real parameters values comparatively to the other modes. This is due to the presence of strong noise in the image, that can and has, in this case, introduced a shift in the mean value

TABLE I: Computed initial AGGMM parameters vs the real parameters in the image of Fig. 2

Mixture parameters	π	μ	α_1	α_2	β
Model design	0.15 0.26 0.59	35 100 180	15 20 18	30 25 10	2 3 1
Computed initial parameters	0.14 0.28 0.58	38 101 179	17.36 21.61 17.60	23.44 24.81 11.38	2.11 2.45 1.13
Δ (computed initial parameters, real parameters)	0.01 0.02 0.02	3 1 1	2.36 1.61 0.40	6.66 0.19 1.38	0.11 0.55 0.13

parameter μ_{init}. This shift occurs especially for flat modes and will impact, in turn, the initialization of the parameter α. Indeed, for Fig. 2, the initialization of $\mu(1)$ has been shifted right by three points which has introduced an error of amount 6.66 in its corresponding right scale parameter α, as we can see in Table.1. Fortunately, this relatively important deviation does not seem to be impacting much the EM execution as it has converged in only 6 iterations.

Afterwards, experiments have been carried out on real images. The proposed initializations are depicted in Fig. 3. We can notice that the initial computed parameters have permitted to initial curves to be close to histograms to be fitted. Furthermore, the final estimated AGGMMs *pdfs* yielded with the initialization provided in [1] and with the ours are depicted in Fig. 4 and Fig. 5. At first sight, our initialization seems to guide the EM algorithm towards the right solutions, resulting in better fitting histograms. In fact, each computed mode is corresponding to a real mode in the image histogram, unlike the other initialization, as illustrated in Fig. 4.B, Fig. 5.C and Fig. 5.D. On the other hand, Fig. 4 allows to appreciate the mixture model based-image segmentation related to each solution. We can see, for example, the effect of a non adequate initialization on the EM convergence on the segmentation in Fig. 4.D, where an internal dark part has not been well revealed as in the segmentation in Fig. 4.E. This under-segmentation is the result of modelling two histogram modes with one *pdf* mode as seen in Fig. 4.B, while the same modes has been modelled by two different *pdfs* modes in Fig. 4.C. Moreover, we can evaluate quantitatively, in Table.3, the results of the solutions given by the two initializations, in the light of the number of iterations needed for EM algorithm convergence and the fitting errors. For all cases, the number of iterations needed when using the initialization in [1], is at least 3 times greater than the one required when using our initialization. This difference between the convergence iterations number in cases of Fig. 2 and Fig. 5.D has risen sharply. This means that the computation load has been alleviated thanks to our computed initial AGGMM parameters, permitting thus to the EM algorithm to achieve quickly the convergence. Concerning the fitting errors, they are relatively similar, as the EM algorithm stops when the fitting curve is very close to the histogram.

TABLE II: Estimated AGGMM parameters of the histogram in Fig. 2 with our initialization VS initialization in [1]

Estimated parameters using initialization in [1]	0.15 0.25 0.60	37.05 99.49 179	15.51 17.90 18.33	28.60 25.40 11.29	1.90 2.77 1.07
Estimated parameters using our initialization	0.15 0.25 0.60	36.19 98.53 179	15.48 17.92 18.31	28.34 25.32 11	1.91 2.71 1.07
Iterations number using initialization in [1]	293				
Iterations number our initialization	**6**				
Fitting error er_{fit} when using initialization in [1]	4×10^{-4}				
Fitting error er_{fit} when using our initialization	**3.8**$\times 10^{-4}$				

TABLE III: Needed iterations number for EM convergence and fitting error for cases A, B and C

Fig. 4.B: Iterations number / initialization by [1]	10
Fig. 4.B: Fitting error / initialization by [1]	10×10^{-4}
Fig. 4.C: Iterations number / our initialization	**3**
Fig. 4.C: Fitting error / our initialization	13×10^{-4}
Fig. 5.B: Iterations number / initialization by [1]	122
Fig. 5.B: Fitting error / initialization by [1]	16×10^{-4}
Fig. 5.C: Iterations number / our initialization	**38**
Fig. 5.C: Fitting error / our initialization	15×10^{-4}
Fig. 5.E: Iterations number / initialization by [1]	616
Fig. 5.E: Fitting error / initialization by [1]	5.77×10^{-4}
Fig. 5.F: Iterations number / our initialization	**43**
Fig. 5.F: Fitting error / our initialization	4×10^{-4}

V. Conclusion

In this work, we have proposed a novel initialization method for EM algorithm to estimate the parameters of an AGGMM which approximates an image histogram. This method is based on geometric computations using the image histogram shape through points selected on the histogram. Experimentations carried out on synthetic and real world images show that the method recovers successfully the input mixture parameters when the implied synthetic image consists in pixels whose gray values are generated with AGGMM. Furthermore, our AGGMM parameters initializations involving real images reveal to be very close to the final parameters obtained for the quoted mixture model. This has lead to correct histogram fitting for synthetic and real world images. In addition to a good histogram mode estimation, our initialization involves, also, less iterations used by the EM algorithm, requiring consequently, less computation time.

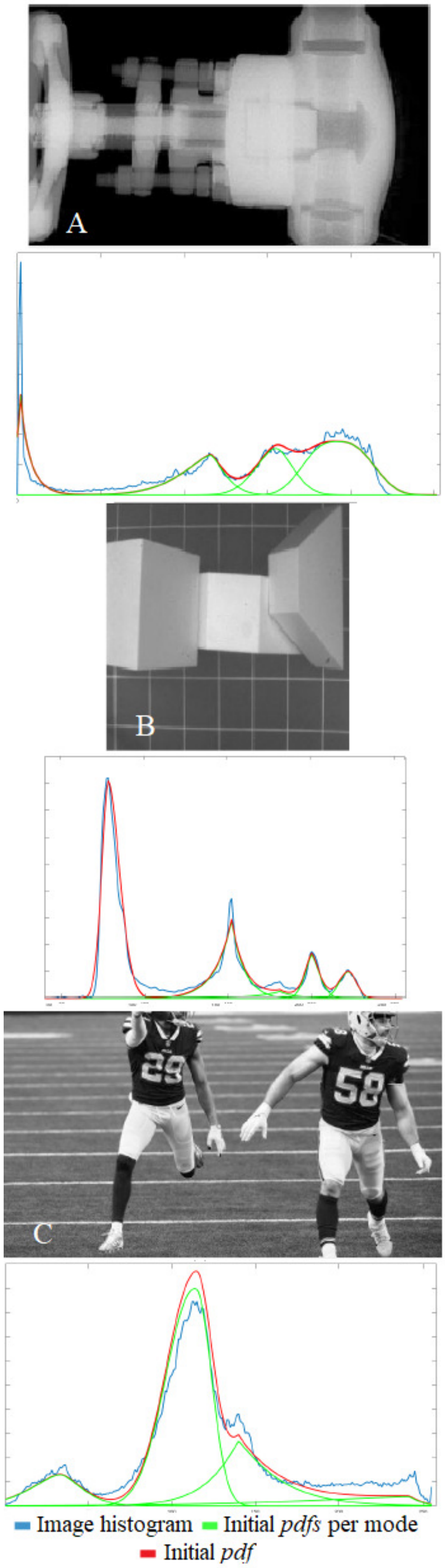

Fig. 3: Real images (A,B and C) followed by their histograms (in blue) and our proposed AGGMM initialization (in red)

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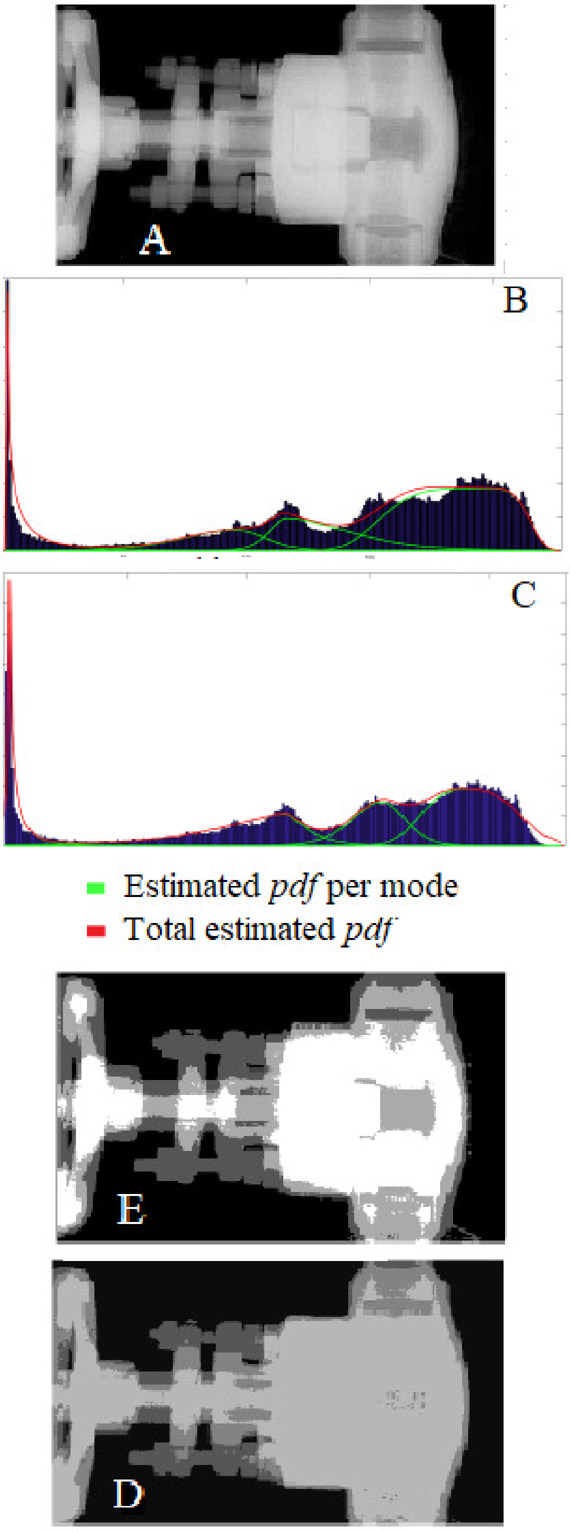

Fig. 4: A: Test image. B: estimated AGGMM *pdf*, (in red) with the initialization provided in [1]. C: Estimated AGGMM *pdf* with our initialization (in red). D: Segmented image on the basis of the AGGMM in B. E: Segmented image on the basis of the AGGMM in C.

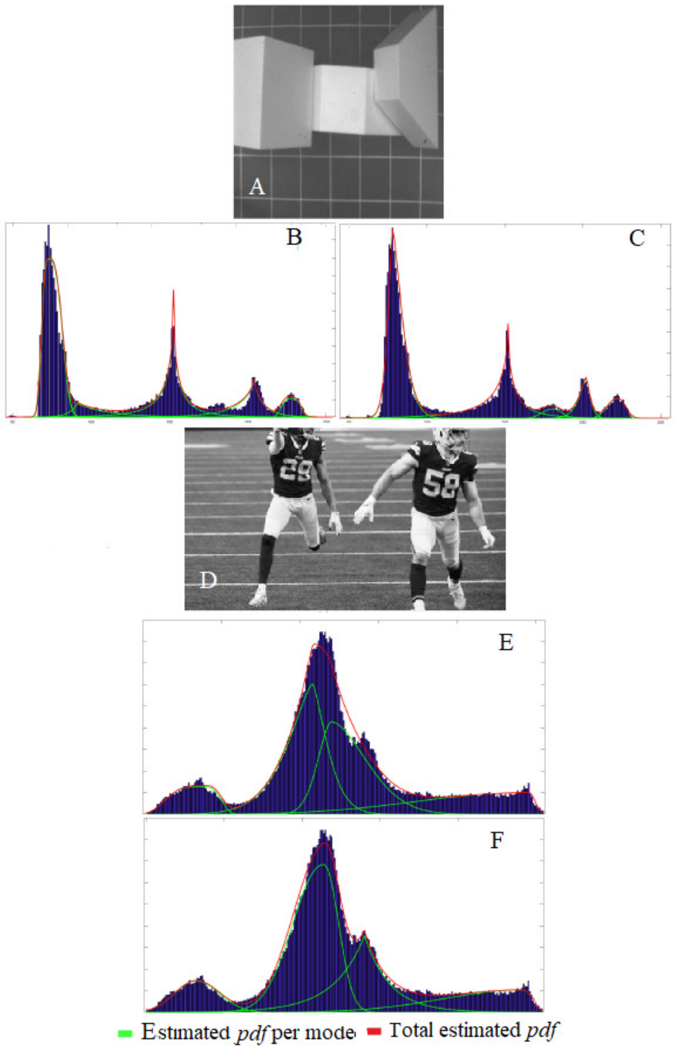

Fig. 5: A and D: Test images. B: Image histogram of A, its EM initialization provided in [1] (in green) and its corresponding final AGGMM *pdf* (in red) . C: Image histogram of A, our proposed EM initialization (in green) and its corresponding final AGGMM *pdf* (in red). E: Image histogram of D, its EM initialization provided in [1] (in green), and its corresponding final AGGMM *pdf* (in red). F: Image histogram of D, our proposed EM initialization (in green), and its corresponding final AGGMM *pdf* (in red)

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